

Sites

Motivation: enlarge the notion of top spaces to arbitrary categories
(to define sheaves, cohomology, etc.)

Def. A Grothendieck topology on a category $\mathcal{C}$ consists of:

for each $X \in \text{Obj}(\mathcal{C})$, $\exists$ set $\text{Cov}(X)$ consisting of coverings of X
 $(\{X_i \rightarrow X\}_{i \in I})$ s.t.

$$(1) \quad \forall X' \rightarrow X \text{ iso} : (X' \rightarrow X) \in \text{Cov}(X)$$

$$(2) \quad \forall \{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X), \quad Y \rightarrow X :$$

$$X_i \times_X Y \text{ exist \& } \{X_i \times_X Y \rightarrow Y\}_{i \in I} \in \text{Cov}(Y)$$

$$(3) \quad \forall \{X_i \rightarrow X\}_{i \in I} \in \text{Cov}(X), \quad \{X_{ij} \rightarrow X_i\}_{j \in J_i} \in \text{Cov}(X_i) :$$

$$\{X_{ij} \rightarrow X_i \rightarrow X\}_{i \in I, j \in J_i} \in \text{Cov}(X)$$

Site = category w/ a Grothendieck topology

Ex. (1) (top spaces) X top space.

$$\text{Op}(X) \text{ w/ } \text{Cov}(U) = \{ \{U_i \rightarrow U\} \mid \bigcup_i U_i = U \}$$

(2) (small étale site) X scheme.

$$\text{category } X_{\text{ét}} \begin{cases} \text{Objects: } U \rightarrow X \text{ étale} \\ \text{Morphisms: } X\text{-morphisms} \end{cases}$$

$$\text{Cov}(U \rightarrow X) = \{ \{U_i \rightarrow U \text{ étale}\} \mid \bigcup_i U_i \rightarrow U \text{ surj.} \}$$

(3) (big étale site $(\text{Sch}_{\text{ét}})$)

$$\text{Sch w/ } \text{Cov}(X) = \{ \{U_i \rightarrow X \text{ étale}\} \mid \bigcup_i U_i \rightarrow X \text{ surj.} \}$$

(similarly, can define Sch_{Zar} , Sch_{Aff} , Sch_{ppc} , $\text{Sch}_{\text{fin-ét}}$)

(4) $\mathcal{C}$ site, $S \in \text{Obj}(\mathcal{C}) \Rightarrow \mathcal{C}/_S$ site

(can define $(\text{Sch}/S)_{\text{ét}}$, $(\text{Sch}/S)_{\text{Zar}}$, $(\text{Sch}/S)_{\text{Aff}}$, $(\text{Sch}/S)_{\text{ppc}}$)

Remark. Ignore set-theoretic issues!

Presheaves and sheaves

Def. A presheaf on a category $\mathcal{C}$ is a contravariant functor $\mathcal{C}^{\text{op}} \rightarrow \text{Set}$.

Def. A sheaf on a site $\mathcal{C}$ is a presheaf $F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ s.t.

$$\forall S \in \text{Obj}(\mathcal{C}), \{S_i \rightarrow S\} \in \text{Cov}(S):$$

$$F(S) \rightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j) \text{ exact.}$$

Prop. F presheaf on Sch . F is a sheaf on $\text{Sch}_{\text{ét}}$ iff

(1) F sheaf on Sch_{Zar}

(2) $\forall S' \rightarrow S$ surj étale morphism of affine schemes:

$$F(S) \rightarrow F(S') \rightrightarrows F(S' \times_S S') \text{ exact.}$$

Prop. Any representable functor on $\text{Sch}/_S$ is a sheaf on $(\text{Sch}/_S)_{\text{ét}}$.

(Slogan: $\{\text{Schemes}\} \subset \{\text{Sheaves}\}$)

pt. Suffices to show $h_X = \text{Mor}_S(-, X): \text{Sch}/_S \rightarrow \text{Set}$ is a sheaf.

(1) is true by gluing of morphisms. (on Zariski open subsets)

For (2), need to show $\forall T' \rightarrow T$ surj étale morphism of affine schemes:

$$\text{Mor}_S(T, X) \rightarrow \text{Mor}_S(T', X) \rightrightarrows \text{Mor}_S(T' \times_T T', X) \text{ exact.}$$

(true by Descent theory!) $\blacksquare$

Def. A morphism of presheaves/sheaves is a natural transformation of functors.

Given $F \rightarrow G, G' \rightarrow G$ morphisms,

define presheaf $F \times_G G': \mathcal{C}^{\text{op}} \rightarrow \text{Set}$

$$(S \mapsto F(S) \times_{G(S)} G'(S))$$

Prop. (a) $F \times_G G'$ fiber product in $\text{Pre}(\mathcal{C}) = \text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$.
(category of presheaves)

(b) If $\mathcal{C}$ site & F, G, G' sheaves, $F \times_G G'$ also sheaf.

In particular, $F \times_G G'$ fiber product in $\text{Sh}(\mathcal{C})$. (category of sheaves)

Theorem. (Sheafification) $\mathcal{C}$ site. $\text{Sh}(\mathcal{C}) \hookrightarrow \text{Pre}(\mathcal{C})$ has a left adjoint $F \mapsto F^{\text{sh}}$.

Remark. We can define sheaves of abelian groups, rings, $\mathcal{O}$ -modules similarly.
 $\mathcal{T}$ (sheaf of rings)

Prestacks

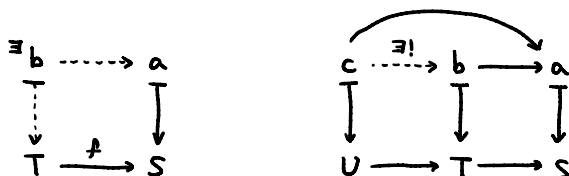
Motivation (from last time): we want $\mathcal{X} \rightarrow \text{Sch}$ s.t.

(a) pullbacks exist (& unique up to iso)

(b) each fiber category $\mathcal{X}(S)$ is a groupoid (automorphism data)

Def. A functor $p: \mathcal{X} \rightarrow \mathcal{C}$ is a prestack over a category $\mathcal{C}$ if

- (1) (pullbacks exist) (2) (universal property of pullbacks)



Obs. ① In the diagram of (1), pullbacks exist uniquely up to unique iso.

(write b as f^*a or $a|_T$ to indicate a choice of a pullback)

② Fiber categories $\mathcal{X}(S)$ are groupoids.

More generally, any $b \rightarrow a$ above $T \xrightarrow{f} S$ is $T \rightarrow S$.

③ After a fixed choice of pullbacks, any map that involves pullbacks and satisfies certain commutativity conditions is uniquely defined.

Ex.
$$\begin{array}{ccc} T' & \rightarrow & T \\ f' \downarrow & \cup & \downarrow f \\ S' & \rightarrow & S \end{array} \quad \xRightarrow{\text{above}} \quad \begin{array}{ccc} f'^*a' & \xrightarrow{\exists!} & f^*a \\ \downarrow & \cup & \downarrow \\ a' & \xrightarrow{\phi} & a \end{array}$$

We denote the unique map $f'^*a' \rightarrow f^*a$ as $\phi|_T$.

Def. (1) A morphism of prestacks over $\mathcal{C}$ is a functor $f: \mathcal{X} \rightarrow \mathcal{Y}$
s.t. $\mathcal{X} \rightarrow \mathcal{Y}$ strictly commutes. (equality of functors)

$$\begin{array}{ccc} & & \\ P_{\mathcal{X}} & \searrow & \swarrow P_{\mathcal{Y}} \\ & \mathcal{C} & \end{array}$$

(2) If $f, g: \mathcal{X} \rightarrow \mathcal{Y}$ are morphisms of prestacks over $\mathcal{C}$,
a 2-morphism $\alpha: f \rightarrow g$ is a natural transformation s.t.

$\forall a \in \text{Obj}(\mathcal{X}): \alpha_a: f(a) \rightarrow g(a)$ in $\mathcal{Y}$ is over $\text{Id}_{P_{\mathcal{X}}(a)}$ in $\mathcal{C}$.

Any 2-morphism is a natural isomorphism.

$$\mathcal{X} \begin{array}{c} \xrightarrow{f} \\ \downarrow \alpha \\ \xrightarrow{g} \end{array} \mathcal{Y}$$

(3) $\text{MOR}(\mathcal{X}, \mathcal{Y})$ category $\left\{ \begin{array}{l} \text{Objects: morphisms of prestacks } \mathcal{X} \rightarrow \mathcal{Y} \\ \text{Morphisms: 2-morphisms} \end{array} \right.$
always a groupoid!

(4) $f: \mathcal{X} \rightarrow \mathcal{Y}$ morphism of prestacks is a $\left\{ \begin{array}{l} \text{monomorphism} \\ \text{epimorphism} \\ \text{isomorphism} \end{array} \right.$
if f is $\left\{ \begin{array}{l} \text{fully faithful} \\ \text{essentially surj.} \\ \text{equiv. of categories} \end{array} \right.$

mono $\Leftrightarrow \forall S \in \text{Obj}(\mathcal{C}): \text{restrictions } f_S: \mathcal{X}(S) \rightarrow \mathcal{Y}(S) \text{ fully faithful}$
iso $\Leftrightarrow \exists g: \mathcal{Y} \rightarrow \mathcal{X}, \text{ 2-isos } g \circ f \xrightarrow{\sim} \text{Id}_{\mathcal{X}}, f \circ g \xrightarrow{\sim} \text{Id}_{\mathcal{Y}}$

Remark. ~~We could define the 2-category of prestacks, but choose not to~~
~~b/c we want to replace identities by isomorphisms in certain notions~~
~~that appear frequently.~~ (Incorrectly stated in lecture)

We will work in the 2-category of prestacks!

A 2-commutative square diagram is a diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow g' & \Downarrow \alpha & \downarrow g \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array} \text{ together w/ a 2-isomorphism } \alpha: g \circ f' \xrightarrow{\sim} f \circ g'.$$

Examples of prestacks

(1) $(\{\text{presheaves}\} \subset \{\text{prestacks}\})$

$F: \mathcal{C}^{\text{op}} \rightarrow \text{Set}$ presheaf.

$\Rightarrow$ prestack χ_F $\left\{ \begin{array}{l} \text{Objects: pairs } (a, S) \text{ s.t. } S \in \text{Obj}(\mathcal{C}), a \in F(S) \\ \text{(over } \mathcal{C}) \downarrow \text{ Morphisms: a map } (a', S') \rightarrow (a, S) \text{ is } f: S' \rightarrow S \\ \text{s.t. } a' = f^* a \end{array} \right.$

$$\chi_F \rightarrow \mathcal{C} \quad ((a, S) \mapsto S)$$

In fact, presheaves = prestacks s.t. $\chi(S)$ are sets

$$\text{Mor}(F, G) \xleftarrow{(\chi_F, \chi_G)} \text{Mor}(\chi_F, \chi_G)$$

(2) $(\{\text{schemes}\} \subset \{\text{prestacks}\})$

(* This example holds for any category, not just Sch)

Apply (1) to schemes. Scheme S

$\Rightarrow$ prestack χ_S $\left\{ \begin{array}{l} \text{Objects: } T \rightarrow S \text{ morphism of schemes (object over } T) \\ \text{(over Sch)} \downarrow \text{ Morphisms: a map } (T' \rightarrow S) \rightarrow (T \rightarrow S) \text{ is } T' \rightarrow T \\ \text{s.t. } \begin{array}{c} T' \rightarrow T \\ \downarrow \chi_S \\ S \end{array} \end{array} \right.$

(3) (prestacks of smooth curves)

$\mathcal{U}$ $\left\{ \begin{array}{l} \text{Objects: family of curves } \mathcal{C} \rightarrow S \text{ (object over } S) \\ \text{(over Sch)} \downarrow \text{ Morphisms: a map } (\mathcal{C}' \rightarrow S') \rightarrow (\mathcal{C} \rightarrow S) \text{ is} \end{array} \right.$

$$a: \mathcal{C}' \rightarrow \mathcal{C}, f: S' \rightarrow S \text{ s.t. } \begin{array}{ccc} \mathcal{C}' & \xrightarrow{a} & \mathcal{C} \\ \downarrow & \square & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

$\mathcal{U}_g =$ full subcategory of $\mathcal{U}$ consisting of family of curves of genus g

(4) (prestacks of coherent sheaves / VBs)

Fix $C/k = \bar{k}$ sm proj conn curve.

$\underline{\text{Coh}}(C)$ $\left\{ \begin{array}{l} \text{Objects: pairs } (E, S) \text{ s.t. } S \in \text{Obj}(\text{Sch}/k), E \in \text{Coh}(C \times_k S) \text{ flat}/S \\ \text{(over Sch}/k) \downarrow \text{ Morphisms: a map } (E', S') \rightarrow (E, S) \text{ is } f: S' \rightarrow S \text{ \& } \end{array} \right.$

$$E \rightarrow (\text{id} \times f)_* E' \text{ whose adjoint is iso}$$

Similarly define subcategories $\text{Bun}(C)$, $\underline{\text{Coh}}_{r,d}(C)$, $\text{Bun}_{r,d}(C)$.

(vector bundles / fixed rank r , degree d)

(5) (Quotient prestacks and stacks)

Def. $G \rightarrow S$ fppf affine group scheme over S .

A principal G -bundle over an S -scheme T is a scheme P

with G -action $\sigma : G \times_S P \rightarrow P$ s.t.

$$\begin{cases} (a) P \rightarrow T \text{ } G\text{-invariant fppf morphism} \\ (b) (\sigma, pr_2) : G \times_S P \rightarrow P \times_T P \quad ((g, p) \mapsto (gp, p)) \text{ is iso.} \end{cases}$$

($\Leftrightarrow \exists$ étale cover $T' \rightarrow T$ s.t. $P \times_T T' \cong G \times_S T'$)

Remark. This notion is a generalization of trivial bundle $G \times_S T \rightarrow T$.

We need it to define an object (stack) that satisfies gluing condition. (Gluing of trivial bundles is not necessarily trivial!)

Setting $G \rightarrow S$ smooth affine group scheme acting on scheme U over S .

Quotient prestack:

$$\begin{aligned} & [U/G]^{\text{pre}} \\ & (\text{over } \text{Sch}/S) \left\{ \begin{array}{l} \text{Objects: } u \in U(T) \text{ (object over } T) \\ \text{Morphisms: a map } (u' \in U(T')) \rightarrow (u \in U(T)) \text{ is} \\ \quad f: T' \rightarrow T \text{ \& } r \in G(T') \text{ s.t. } u' = r \cdot (u \circ f). \end{array} \right. \end{aligned}$$

Quotient stack:

$$\begin{aligned} & [U/G] \\ & (\text{over } \text{Sch}/S) \left\{ \begin{array}{l} \text{Objects: } \begin{array}{c} P \rightarrow U \\ \downarrow \\ T \end{array} \text{ where } P \rightarrow T \text{ principal } G\text{-bundle} \\ \quad \quad \quad \& P \rightarrow U \text{ } G\text{-equivariant morphism} \\ \quad \quad \quad \text{(object over } T) \\ \text{Morphisms: a map } \begin{pmatrix} P' \rightarrow U \\ \downarrow \\ T' \end{pmatrix} \rightarrow \begin{pmatrix} P \rightarrow U \\ \downarrow \\ T \end{pmatrix} \text{ is} \\ \quad \quad \quad T' \rightarrow T \text{ \& } G\text{-equivariant morphism } P' \rightarrow P \\ \quad \quad \quad \text{s.t.} \end{array} \right. \end{aligned}$$

$$\begin{array}{ccccc} & & \xrightarrow{G} & & \\ P' & \longrightarrow & P & \longrightarrow & U \\ \downarrow & \square & \downarrow & & \\ T' & \longrightarrow & T & & \end{array}$$

(6) (Classifying stack)

$G \rightarrow S$ smooth affine group scheme.

BG (over Sch/S) $\left\{ \begin{array}{l} \text{Objects: principal } G\text{-bundles } P \rightarrow T \text{ (object over } T\text{)} \\ \text{Morphisms: a map } (P' \rightarrow T') \rightarrow (P \rightarrow T) \text{ is } T' \rightarrow T \text{ \& } \\ \quad G\text{-equivariant morphism } P' \rightarrow P \text{ s.t. } \begin{array}{ccc} P' & \rightarrow & P \\ \downarrow & \square & \downarrow \\ T' & \rightarrow & T \end{array} \end{array} \right.$

Remark. This is a special case of quotient stack when $U = S$.

The 2-Yoneda Lemma

Lemma. $\mathcal{X}$ prestack over category $\mathcal{C}$, and $S \in \text{Obj}(\mathcal{C})$.

The functor $\text{MOR}(S, \mathcal{X}) \rightarrow \mathcal{X}(S)$ ($f \mapsto f(\text{id}_S)$)

is an equivalence of categories.

pf. Need to show that it is fully faithful and essentially surjective.

(fully faithful) Need to show $\text{Mor}(f, g) \rightarrow \text{Mor}(f(\text{id}_S), g(\text{id}_S))$ bij.

(inj.) Suppose $\phi, \psi: f \rightarrow g$ satisfies $\phi_{\text{id}_S} = \psi_{\text{id}_S} = \alpha$.

$$\forall (h: T \rightarrow S) \in \text{Obj}(S):$$

$$\begin{array}{ccccc} & & \alpha = f(h \rightarrow \text{id}_S) & & \\ & \searrow & & \nearrow & \\ f(h) & \overset{\exists!}{\dashrightarrow} & g(h) & \longrightarrow & g(\text{id}_S) \\ \downarrow & & \downarrow & & \downarrow \\ T & \xrightarrow{\text{id}_T} & T & \xrightarrow{h} & S \end{array}$$

By uniqueness, $\phi_h = \psi_h$.

(surj.) For any $\alpha \in \text{Mor}(f(\text{id}_S), g(\text{id}_S))$, we can use same diagram to get maps $\phi_h: f(h) \rightarrow g(h)$, which glues up to $\phi: f \rightarrow g$ & $\phi \mapsto \alpha$.

(essentially surjective) Take any $a \in \mathcal{X}(S)$ and choose pullbacks h^*a

for all $(h: T \rightarrow S) \in \text{Obj}(\mathcal{S})$.

We can define the morphism $f: S \rightarrow \mathcal{X}$ as

- $(h: T \rightarrow S) \mapsto h^*a$
- $\left((T' \xrightarrow{h'} S) \xrightarrow{k} (T \xrightarrow{h} S) \right) \mapsto (h'^*a \rightarrow h^*a)$ defined by

$$\begin{array}{ccccc}
 & & \xrightarrow{\exists!} & & \\
 h'^*a & \cdots \rightarrow & h^*a & \rightarrow & a \\
 \downarrow & & \downarrow & & \downarrow \\
 T' & \xrightarrow{k} & T & \xrightarrow{h} & S
 \end{array}$$

Then $\text{MOR}(S, \mathcal{X}) \rightarrow \mathcal{X}(S)$

$$f \mapsto (\text{id}_S)^*a \cong a.$$



Exercise. If $\mathcal{X}$ is a presheaf, both $\text{MOR}(S, \mathcal{X})$ and $\mathcal{X}(S)$ are sets and the statement recovers the Yoneda Lemma.